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1986 J. Phys. A: Math. Gen. 19 873

(http://iopscience.iop.org/0305-4470/19/6/018)

J. Phys. A: Math. Gen. 19 (1986) 873-885. Printed in Great Britain

Diabolical points in one-dimensional Hamiltonians quartic in the momentum

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Received 5 August 1985, in final form 24 September 1985

Abstract. We study the family of quantal Hamiltonians defined on $-1<x<+1$ by

$$\hat{H}=p^4=\mathrm{d}^4/\mathrm{d}x^4 \qquad \text{and} \qquad \begin{pmatrix}\psi''(\pm1)\\ \psi'''(\pm1)\end{pmatrix}=\begin{pmatrix}c+\mathrm{i}d & \pm a\\ \pm b & -c+\mathrm{i}d\end{pmatrix}\begin{pmatrix}\psi(\pm1)\\ \psi'(\pm1)\end{pmatrix}$$

which depend via the boundary conditions on four real parameters a, b, c, d. The spectra of states with even and odd parity have degeneracies with codimension three, that is, on lines in $abcd$ space; near the degeneracies, the level splitting has a double-cone (diabolical) structure, and during a circuit of a diabolical point in the subspace of real $\hat{H}$ (i.e. $d=0$) the wavefunctions change sign. These are generic properties which in this model can be studied analytically. The degeneracies could be realised classically in vibrating beams with positive and negative linear feedback in the boundary conditions.

1. Introduction

Almost all stationary bound quantal systems have non-degenerate energy levels. It is however possible to produce degeneracies (of the type usually called 'accidental') by varying several parameters in the Hamiltonian $\hat{H}$. Von Neumann and Wigner (1929) showed that two parameters are required if $\hat{H}$ is real symmetric, and three if $\hat{H}$ is complex Hermitian. Teller (1937) showed that in the space of energy and parameters the energy level surfaces near the degeneracy have the form of a double cone, that is, a diabolo. Therefore degenerate Hamiltonians can be described as diabolical points (Berry 1983) in parameter space.

This generic structure of degeneracies in quantum mechanics cannot be illustrated by one-dimensional Hamiltonians of the familiar form $p^2+V(x)$ with the solutions of Schrödinger's equation restricted by boundary conditions at two points, because degeneracies are absolutely forbidden in such systems. The reason is that for given energy E the solution satisfying one boundary condition is unique and either does satisfy the other boundary condition (i.e. E is an eigenvalue) or does not (i.e. E is not an eigenvalue). Therefore it was thought necessary to increase the dimensionality in order to find degeneracies, and indeed by numerical computation Berry and Wilkinson (1984) discovered diabolical points in the spectra of planar quantum triangles (vibrating triangular membranes).

To obtain a more complete understanding of degeneracies, it is desirable to supplement genericity arguments and numerical exploration with a model for which exact analytical results can be obtained, and this is our present purpose. We will show that a suitable alternative to increasing the dimensionality from one to two is to remain in one dimension and increase the order of $\hat{H}$ from p^2 to p^4. In quantum mechanics such higher-order dependence can rise from p truncations of effective band Hamiltonians for electrons in crystals with magnetic fields. There are several ways to introduce parameters $\{a_i\}$, for example via a potential $V(x; \{a_i\})$, but we find it simpler to consider the system as free ($V=0$) on the interval $-1 \leq x \leq 1$ and introduce parameters into the boundary conditions at ± 1. In this way we will construct (§ 2) a four-parameter family of Hamiltonian operators $\hat{H}(\{a_i\})$.

In § 3 we determine the degeneracies analytically, confirm that their codimensions accord with the Von Neumann–Wigner theorem in both the real and complex cases, and demonstrate the diabolical structure of the energy level surfaces. In § 4 we find pairs of degenerate states and illustrate the sign change of states during a circuit of a degeneracy. Finally (§ 5) we show that these degenerate quartic Hamiltonians can be realised physically as vibrating beams with unusual boundary conditions.

2. Quartic Hamiltonians

When $\hat{H} = p^4$ the Schrödinger equation is (denoting derivatives by primes)

$$\psi'''' = E\psi \qquad |x| \leq 1 \tag{1}$$

with boundary conditions chosen to ensure Hermiticity, that is

$$\langle\phi|\hat{H}|\psi\rangle = \int_{-1}^{1} \mathrm{d}x\, \phi^*(x)\psi''''(x) = \int_{-1}^{1} \mathrm{d}x\, \psi(x)\phi^{*''''}(x) = \langle\psi|\hat{H}|\phi\rangle^* \tag{2}$$

where ϕ and ψ are any functions in the Hilbert space. Integration by parts gives

$$G(+1) = G(-1) \tag{3}$$

where

$$G(x) = \phi^*\psi''' - \phi^{*\prime}\psi'' + \phi^{*\prime\prime}\psi' - \phi^{*\prime\prime\prime}\psi. \tag{4}$$

If the boundary conditions are to be applied separately to $x = \pm 1$ (that is, if periodic boundary conditions are excluded), then $G(+1)$ and $G(-1)$ must vanish separately. Because ϕ and ψ are independent this implies at least two relations between derivatives, which it is convenient to write as

$$\begin{pmatrix} \psi'' \\ \psi''' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \qquad \text{at } x = \pm 1 \tag{5}$$

with M_{ij} complex. Direct subsitution into (4) leads to

$$M_{22} = -M_{11}^* \qquad M_{12} = M_{12}^* \qquad M_{21} = M_{21}^*. \tag{6}$$

These relations can be satisfied by different M_{ij} at the two endpoints. However, by choosing M_{ij} to make $\hat{H}$ symmetric under reflection in $x=0$ we will produce a spectrum partitioned into states with even parity and states with odd parity, and we will be able to find diabolical points for each of these spectra, i.e. degeneracies between states of the same symmetry class. All such even Hamiltonians are generated by the following boundary conditions (which satisfy (6))

$$\begin{pmatrix} \psi''(\pm 1) \\ \psi'''(\pm 1) \end{pmatrix} = \begin{pmatrix} c+\mathrm{i}d & \pm a \\ \pm b & -c+\mathrm{i}d \end{pmatrix} \begin{pmatrix} \psi(\pm 1) \\ \psi'(\pm 1) \end{pmatrix} \tag{7}$$

in which a, b, c, d are real.

Together with (1), these conditions define our four-parameter family of quartic Hamiltonians $\hat{H}(\{a_i\})$ with parameters $\{a_i\}=(a, b, c, d)$. When $d \neq 0$, $\hat{H}$ is complex Hermitian; when $d=0$, $\hat{H}$ is real symmetric.

3. Diabolical points

For the even and odd states it will be natural to consider positive and negative energies separately, i.e. four cases in all. For each, the levels will be determined by an eigenvalue condition

$$f(E;\{a_i\})=0. \tag{8}$$

For almost all $\{a_i\}$, f has simple zeros at eigenvalues E, but at diabolical points $\{a_i^*\}$ the zeros E^* are (at least) double, so that in addition to (8) we must have

$$f_E(E;\{a_i^*\})=0. \tag{9}$$

If f were a generic function, its real zeros E would coalesce and disappear as the parameters passed through diabolical values. But the eigenvalues of Hermitian operators are continuous functions of parameters and cannot disappear, so that f is not a generic function. Indeed, from the fact that the rate of change of eigenvalue with parameters, namely

$$\partial E/\partial a_i = -f_{a_i}/f_E \tag{10}$$

must remain finite, even at $\{a_i^*\}$, it follows from (9) that

$$f_{a_i}\{E^*,\{a_i^*\})=0 \tag{11}$$

for each of the parameters a_i. These relations greatly simplify the determination of diabolical points.

For *positive energies*, the *even* solutions of (1) are

$$\psi(x)=\mu \cos kx + \nu \cosh kx \tag{12}$$

with

$$E \equiv k^4. \tag{13}$$

Eigenvalues k, and the coefficients μ and ν, can be determined from the boundary conditions (7). It is convenient to define rescaled parameters

$$\{A_i\}=(A, B, C, D)\equiv(ak^{-1}, bk^{-3}, ck^{-2}, dk^{-2}). \tag{14}$$

Direct substitution of (12) into (7) leads to the eigenvalue condition

$$F(k;\{A_i\})=-(T+t)(AB+C^2+D^2+1)+2tTA+2B+2(t-T)C=0 \tag{15}$$

where

$$T\equiv \tanh k \qquad t\equiv \tan k. \tag{16}$$

From the t periodicity we see that the eigenvalues form hypersurfaces in k, A, B, C, D space with k separation approximately given by π. It is necessary to consider only positive k.

The condition (15), involving the scaled variables, has the same form as (8), so that (using (11)) the diabolical points are $\{A_i^*\}$, determined by

$$F_k(k^*;\{A_i^*\})=0. \tag{17}$$

It is more convenient to use (11) directly, thereby giving the degeneracies parametrically as functions of k^*. The results are

$$\begin{aligned} A&=2/(T+t)\\ B&=2tT/(T+t)\\ C&=-(T-t)/(T+t)\\ D&=0 \end{aligned} \tag{18}$$

where here and henceforth we abandon the * notation whenever this will not cause confusion. Direct substitution confirms that (15) and (16) are satisfied.

Equations (18) show that for every value of k there is one set of diabolical parameters for which $\hat{H}$ has the degenerate eigenvalue k. As k increases from zero to infinity the diabolical points trace out a series of infinite lines in the four-dimensional parameter space *ABCD*, so that the degeneracies do indeed have the expected codimension for families of complex Hermitian operators, namely three. All the degenerate Hamiltonians have $D=0$, so that the infinite lines of diabolical points lie entirely in the *ABC* subspace and therefore do indeed have the expected codimension for families of real symmetric operators, namely two. Figure 1 shows the first two diabolical lines in the original *abc* space.

Now we exhibit explicitly the diabolical structure of the eigensurfaces close to degeneracies. Choose one of the diabolical lines and fix the value of C^* and thence that of k^*, A^*, B^*, D^*. Now move off the line, to parameters

$$A=A^*+\Delta A \qquad B=B^*+\Delta B \qquad D=D^*+\Delta D. \tag{19}$$

This will lift the degeneracy and give eigenvalues

$$k=k^*+\Delta k^\pm \tag{20}$$

which can be determined by expanding (15) about $\{A_i^*\}$. Because of (17) and (11) it is necessary to go to second order, leading to the result

$$\Delta k^\pm = F_{kk}^{-1}\left\{-\sum_{\mu=1}^{3} F_{k\mu}\,\Delta A_\mu \pm\left[\left(\sum_{\mu=1}^{3} F_{k\mu}\,\Delta A_\mu\right)^2-\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} F_{\mu\nu}\,\Delta A_\mu\,\Delta A_\nu\right]^{1/2}\right\} \tag{21}$$

where μ and ν refer to A, B and D and all derivatives are evaluated at $\{A_i^*\}$, k^*. Evaluating the derivatives leads to

$$\begin{aligned}\Delta k^\pm = {}&[4(1+t^2)(1-T^2)]^{-1}[\![-(t^2+T^2)\,\Delta A+(2+t^2-T^2)\,\Delta B\\ &\pm\{(1-X)[(T^2+t^2)\,\Delta A-(2+t^2-T^2)\,\Delta B]^2\\ &+X[(T^2+t^2)\,\Delta A+(2+t^2-T^2)\,\Delta B]^2\\ &+4(1+t^2)(1-T^2)(T+t)^2(\Delta D)^2\}^{1/2}]\!]\end{aligned} \tag{22}$$

where

$$X\equiv(1+t^2)(1-T^2)(T+t)^2/[(2+t^2-T^2)(T^2+t^2)]. \tag{23}$$

Now $0\leqslant X\leqslant 1$, so that the level splitting is given by the square root of the sum of three squares and the eigenvalues are therefore (hyper)conical in *ABCDk* space.

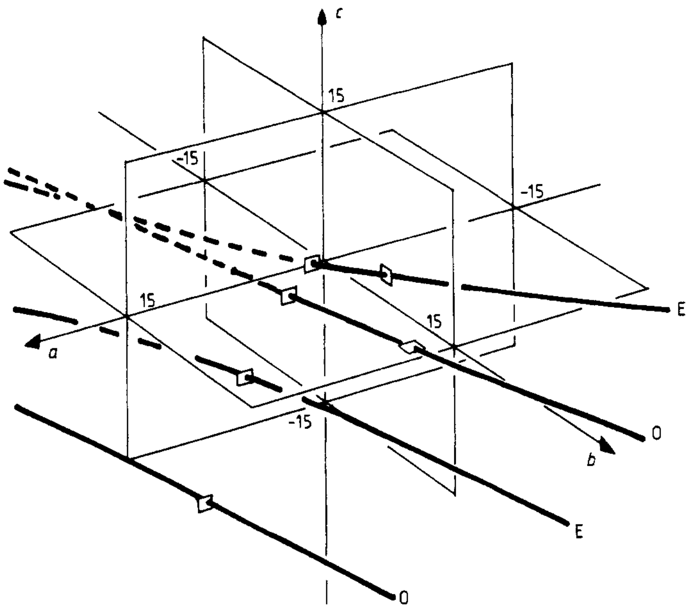

Figure 1. Lowest two diabolical lines for even (E) and odd (O) states, in parameter space of real Hamiltonians; the negative-energy parts of the lowest lines are shown broken.

It is instructive to consider the Hamiltonians with $C = D = 0$, i.e. the two-parameter family of real operators $\hat{H}(A, B)$. From (18), the degenerate eigenvalues k_j are the solutions of

$$T_j \equiv \tanh k_j = \tan k_j \qquad \text{i.e. } k_j \approx (j+\tfrac{1}{4})\pi - 2\mathrm{e}_j \qquad (j = 1, 2 \ldots) \tag{24}$$

where

$$\mathrm{e}_j \equiv \exp[2\pi(j+\tfrac{1}{4})] \tag{25}$$

and the diabolical points in the AB plane lie at

$$\begin{aligned} A_j &= 1/T_j \approx 1 + 2/\mathrm{e}_j \\ B_j &= T_j \approx 1 - 2/\mathrm{e}_j. \end{aligned} \tag{26}$$

These all lie extremely close to $A = B = 1$; for example $A_1 = 1.000\,77$ and $A_2 = 1.000\,001\,45$. Figure 2 shows the lowest levels $k_j(A)$ for $B = B_1$, illustrating the fact that their approach and degeneration occur in a very small parameter region, outside which they vary slowly.

For $C = D = 0$ the cones (22) become

$$\begin{aligned} \Delta k_j^{\pm} &= \{-T_j^2\,\Delta A + \Delta B \pm [T_j^4(T_j^2\,\Delta A - \Delta B)^2 + (1 - T_j^4)(T_j^2\,\Delta A + \Delta B)^2]^{1/2}\}/[2(1 - T_j^4)] \\ &\approx \tfrac{1}{16}\mathrm{e}_j\{\Delta B - \Delta A \pm [(\Delta A - \Delta B)^2 + (8/\mathrm{e}_j)(\Delta A + \Delta B)^2]^{1/2}\}. \end{aligned} \tag{27}$$

Because T_j is very close to unity (i.e. e_j are large), the cones are extremely elongated in the direction $\Delta A = \Delta B$, their sections with constant splitting being ellipses with axial ratios close to $\sqrt{(2/\mathrm{e}_j)}$. Figure 3 illustrates this.

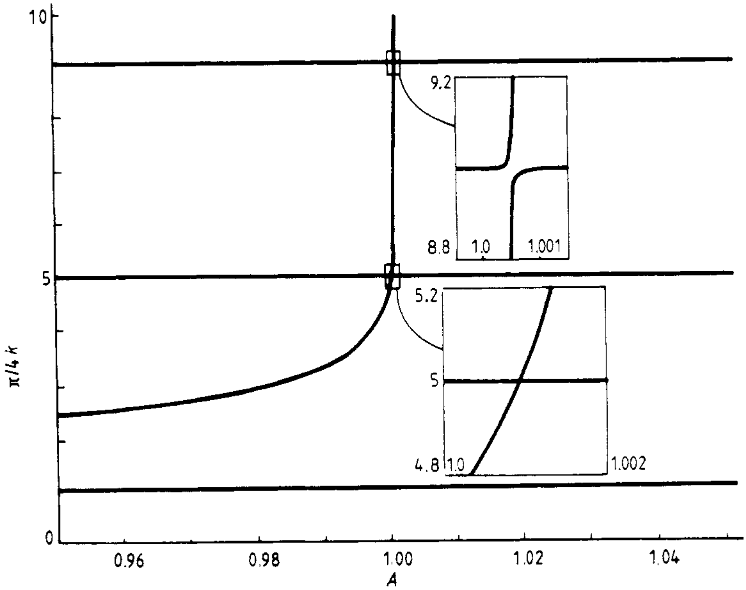

Figure 2. Levels k_i of even states as functions of parameter A, with B fixed at its diabolical value B_1 and $C = D = 0$.

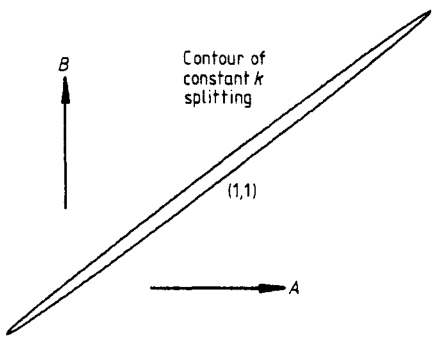

Figure 3. Conical structure near lowest even-parity diabolical point when $C = D = 0$.

These results can be expressed in terms of the physical parameters $\{a_i\}$ by using (14). The only subtlety is that the cones are sheared with respect to those given by (22) and (26), because away from $\{a_i^*\}$ a fixed set of $\{A_i\}$ corresponds to different values of $\{a_i\}$ on each of the two sheets of the diabolo (which have different k values).

For *negative energies*, the *even* solutions can be obtained from (12) by defining

$$E \equiv -\kappa^4 \qquad \text{i.e. } k = \kappa\, e^{i\pi/4} \tag{28}$$

and scaled parameters (cf 14) by

$$\{\alpha_i\} = (\alpha, \beta, \gamma, \delta) \equiv (a\kappa^{-1}, b\kappa^{-3}, c\kappa^{-2}, d\kappa^{-2})$$

i.e.

$$\alpha = A\, e^{i\pi/4} \qquad \beta = B\, e^{3i\pi/4} \qquad \gamma = C\, e^{i\pi/2} \qquad \delta = D\, e^{i\pi/2}. \tag{29}$$

It is also convenient to define, instead of (16)

$$SH \equiv \sinh(\kappa\sqrt{2}) \qquad CH \equiv \cosh(\kappa\sqrt{2}) \qquad ss \equiv \sin(\kappa\sqrt{2}) \qquad cc \equiv \cos(\kappa\sqrt{2}). \tag{30}$$

Then the diabolical line expressed parametrically in terms of κ can be obtained from (18) by direct substitution:

$$\begin{aligned} \alpha &= \sqrt{2}(CH + cs)/(SH + ss) \\ \beta &= -\sqrt{2}(CH - cs)/(SH + ss) \\ \gamma &= -(SH - ss)/(SH + ss) \\ \delta &= 0. \end{aligned} \tag{31}$$

As for positive energies, all diabolical points lie in the subspace a, b, c of real operators.

As κ^* increases from zero to infinity the parameters trace out a single diabolical line which joins smoothly (figure 1) with the lowest of the positive-energy ones at $E^* = 0$ (for which $a = 1$, $b = c = 0$). For all these negative-energy diabolical points, the degeneracy occurs for the *ground state* of $\hat{H}$. On the other hand, when $E^* > 0$ the degeneracies occur for excited states. This difference does not imply a higher-order (e.g. triple) degeneracy at $E^* = 0$, because the extra levels come from $E = -\infty$. This is clear from figure 4, which shows the levels E for the degenerate Hamiltonians with parameters $\{a_i^*\} = \{a_i(E^*)\}$.

The *odd* solutions of (1) for *positive energy* are

$$\psi(x) = \mu \sin kx + \nu \sinh kx \tag{32}$$

and arguments precisely similar to those employed in the even case lead to the following diabolical lines, analogous to (18):

$$\begin{aligned} A &= 2tT/(t - T) \\ B &= -2/(t - T) \\ C &= -(t + T)/(t - T) \\ D &= 0. \end{aligned} \tag{33}$$

The lines are qualitatively similar to those for the even case, as discussed following (18); the lowext two are shown in figure 1.

The *odd* solutions for *negative* energy can be found by the substitutions (29) and (30), and the single diabolical line, analogous to (31), is

$$\begin{aligned} \alpha &= \sqrt{2}(CH - cs)/(SH - ss) \\ \beta &= -\sqrt{2}(CH + cs)/(SH - ss) \\ \gamma &= -(SH + ss)/(SH - ss) \\ \delta &= 0. \end{aligned} \tag{34}$$

As for the even solutions, this line connects with the lowest of (33) at $E = 0$, this time at $a = -b = -c = 3$.

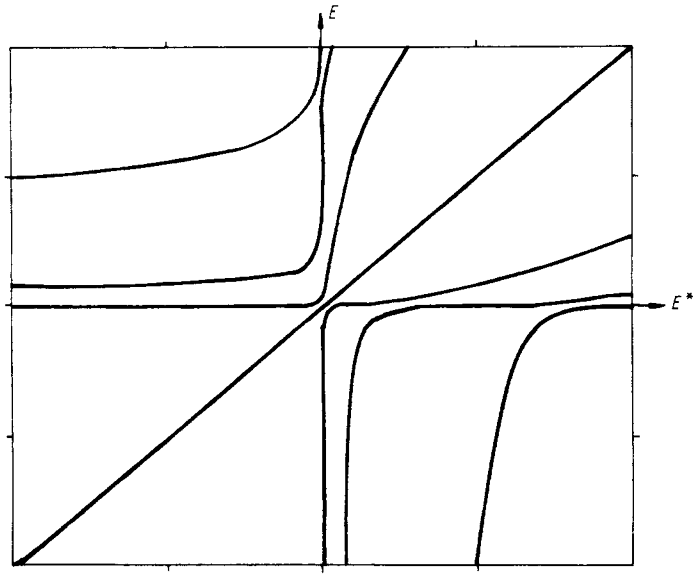

Figure 4. Even states E of Hamiltonians on the diabolical lines, as a function of the energy E^* at which two of the states are degenerate for $-20\,000 < E$, $E^* < +20\,000$. The corresponding parameters $\{a_i(E^*)\}$ are given by equations (14) and (18) (for $E^* > 0$) and (29) and (31) (for $E^* < 0$). Note that the degeneracy line $E = E^*$ is isolated from the other states (no higher-order degeneracies), and that the figure is symmetric about $E = E^*$.

Finally, we note that all diabolical points, even or odd, with positive or negative energies, satisfy

$$c^2 + ab = E. \tag{35}$$

4. States close to degeneracies

Here we consider only the even solutions; results for odd solutions are similar. We begin by studying the pairs of degenerate states on the diabolical lines. Pure cosine solutions ($\nu = 0$ in (12)) of (1) satisfy boundary conditions (7) provided

$$\begin{aligned} c - atk &= -k^2 \\ b + ktc &= k^3 t \\ d &= 0 \end{aligned} \tag{36}$$

where $t \equiv \tan k$. Eliminating t leads to (35), showing that pure cosine solutions satisfy the boundary conditions on the diabolical lines. Similarly, pure cosh solutions ($\mu = 0$ in (12)) are also possible on the diabolical lines.

The degenerate cos and cosh solutions are not orthogonal for finite parameter values, but it is easy to construct linear combinations which are. One such orthogonal normalised pair for the diabolical points labelled j in the ab plane (i.e. $c = 0$, equations

(24)–(26)) is

$$\psi_{1j}=\frac{S_j\cos k_jx-s_j\cosh k_jx}{(s_j^2+S_j^2-2s_j^2S_j^2/k_jT_j)^{1/2}}$$
$$\psi_{2j}=\frac{(-1)^{j+1}[s_j(1-S_j^2/k_jT_j)\cos k_jx+S_j(1-s_j^2/k_jT_j)\cosh k_jx]}{[s_j^2+S_j^2+(s_j^4+S_j^4)/(k_jT_j)-(5/k_j^2T_j^2)(s_j^2+S_j^2)s_j^2S_j^2+(6s_j^4S_j^4)/(k_j^3T_j^3)]^{1/2}} \tag{37}$$

where

$$s_j\equiv\sin k_j \qquad S_j\equiv\sinh k_j. \tag{38}$$

Close approximations (cf (25) and (26)) are

$$\psi_{1j}\approx\frac{\cos k_jx-(-1)^j\surd(2/\mathrm{e}_j)\cosh k_jx}{(1-1/k_j)^{1/2}}$$
$$\psi_{2j}\approx\frac{(1/\surd 2)\cos k_jx-(2/\surd\mathrm{e}_j)(-1)^j(k_j-\frac{1}{2})\cosh k_jx}{[k_j(1-5/2k_j+3/2k_j^2)]^{1/2}}. \tag{39}$$

These pairs of degenerate wavefunctions are shown in figure 5. It is evident that the states ψ_{1j} oscillate with constant amplitude and a frequency that increases slightly near $x=1$ as the cosh term becomes appreciable. The other states ψ_{2j} are concentrated near the boundary where their amplitude is large ($\sim\surd k$); away from $x=1$ they have much smaller amplitude ($\sim 1/\sqrt{2k}$) and oscillate in phase with ψ_{1j}. There is a semiclassical interpretation of the fact that some states are localised and others are not. The classical equation $E=p^4$ has solutions for purely real and purely imaginary momenta p. The former correspond to de Broglie waves and the latter to evanescent waves clinging to the boundary. For almost all parameter values all the states are well approximated by de Broglie waves and the evanescent waves make only a small

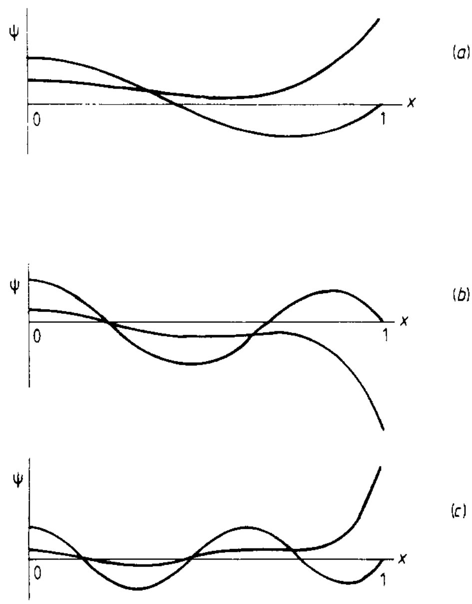

Figure 5. Pairs of degenerate wavefunctions for even states with $C=D=0$: (a) $j=1$, (b) $j=2$, (c) $j=3$.

contribution, as in ψ_{1j}. Indeed, Bohr-Sommerfeld (or its equivalent, Weyl) quantisation (Berry 1983), which is based entirely on real p waves, predicts for the even and odd spectra the asymptotic k separation π—precisely concordant with the exact quantisation formula (15) and its odd-parity equivalent. Exceptionally, however, waves which are entirely or almost entirely evanescent can satisfy the boundary conditions and form eigenstates, as in ψ_{2j}.

During a circuit in parameter space close to a degeneracy, eigenstates are not single-valued but acquire a phase factor. In the general case (Berry 1984) of circuits involving complex Hermitian Hamiltonians the phase may take any value. In the special case of circuits involving only real Hamiltonians, the phase is π (corresponding to a sign change of the eigenstate) if the circuit contains the degeneracy, and zero if it does not (Herzberg and Longuet-Higgins 1963). Here we illustrate the real case for our quartic Hamiltonian with parameters $c = d = 0$ by calculating the coefficient μ in (12) during a circuit of a degeneracy in the AB plane.

From the boundary conditions it follows that

$$\frac{\mu}{\nu}=\frac{\cosh k}{\cos k}\left(\frac{T-B}{B-t}\right). \tag{40}$$

Expanding near the jth diabolical point (24)-(26) using the notations (19) and (20) gives

$$\frac{\mu}{\nu}\approx\left(\frac{1+T_j^2}{1-T_j^2}\right)^{1/2}\left(\frac{\Delta k^{\pm}(1-T_j^2)-\Delta B}{\Delta B-\Delta k^{\pm}(1+T_j^2)}\right) \tag{41}$$

with $\Delta k^{\pm}$ given by the cone formulae (27). Normalisation of (12) gives, using the notation (38)

$$1=\mu^2\left(1+\frac{s_j^2}{k_jT_j}\right)+\nu^2\left(1+\frac{S_j^2}{k_jT_j}\right)+\frac{4\mu\nu s_jS_j}{k_jT_j} \tag{42}$$

and this, together with (41), determines μ as a function of ΔA and ΔB, i.e. near the degeneracy. Now consider a small circuit parametrised by θ, i.e.

$$\Delta A=\Delta\cos\theta \qquad \Delta B=\Delta\sin\theta \qquad (0\leqslant\theta\leqslant 2\pi). \tag{43}$$

then $\mu(\theta)$ varies smoothly and vanishes only when the numerator of (41) does. For the positive sheet Δk^{+} of the cone this happens only when

$$\Delta A=-\Delta B/T_j^2 \qquad \Delta B>0 \qquad \text{i.e. } \theta\approx 3\pi/4. \tag{44}$$

Therefore during a circuit of the diabolical point there is indeed only one sign change for $\mu(\theta)$ (and also one for $\nu(\theta)$, at $\theta\approx 7\pi/4$). For the negative sheet Δk^{-}, the behaviour is the same except for a π rotation, i.e. μ vanishes near $\theta = 7\pi/4$, and ν vanishes near $\theta = 3\pi/4$.

Figure 6 shows $\mu(\theta)$ for circuits of the first two diabolical points. As expected, μ changes sign near $\theta = 3\pi/4$, but what is surprising is the high magnification (100 for $j = 1$, 50 000 for $j = 2$) required to reveal this, arising from the fact that $\mu(\theta)$ is very close to zero over a large angular range. This is a consequence of the cones' high eccentricity (figure 3)—a phenomenon already discovered for diabolical triangles by Berry and Wilkinson (1984).

At least one member of any pair of orthogonal real states must have a zero. In particular this holds near the diabolical points with negative energy (equations (31) and (34)) where it implies the existence of Hamiltonians whose *ground state* has a

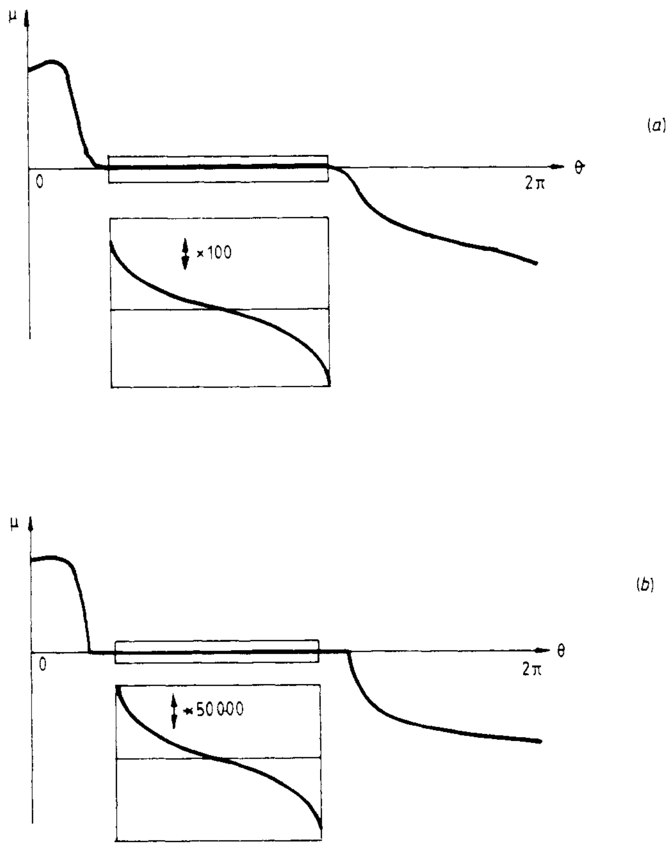

Figure 6. Coefficient $\mu(\theta)$ for circuit of diabolical points with $c=d=0$ and (a) $j=1$, (b) $j=2$.

zero (nodal lines in the fundamental mode of a vibrating plate were discovered by Duffin (1962, 1974)).

5. Diabolical vibrating beams

Small displacements $y(x, t)$ of a vibrating beam satisfy the biharmonic wave equation (Landau and Lifshitz 1959), namely

$$\rho\, \partial^2 y/\partial t^2 = -YI\, \partial^4 y/\partial x^4 \tag{45}$$

where ρ is the linear density, Y is Young's modulus and I is the second moment of area. Therefore harmonic vibrations

$$y(x, t) = \psi(x) \cos \omega t \tag{46}$$

satisfy (1), with the frequency given by

$$\omega^2 = (YIE/\rho l^4) \tag{47}$$

where l is the length of the beam.

Now suppose the beam is clamped horizontally at $x = 0$. Then the vibrations are the positive-energy, even-parity solutions of (1), which depend on the boundary conditions at $x = l$. These consist of two relations between the following four quantities:

$$\begin{aligned} &\text{displacement:} && \psi(l) \\ &\text{slope:} && \psi'(l) \\ &\text{torque:} && YI\psi''(l) \\ &\text{shear force:} && -YI\psi'''(l). \end{aligned} \tag{48}$$

Let these relations be given by (7) with $c = d = 0$, that is (after scaling by l)

$$\psi''(l) = (a/l)\psi'(l) \qquad \psi'''(l) = (b/l^3)\psi(l). \tag{49}$$

Now (24), (26) and (14) show that degenerate vibration modes occur when the physical boundary conditions are

$$\frac{\text{torque}}{\text{slope}} = \frac{k_j YI}{lT_j} \qquad \text{and} \qquad \frac{\text{shear force}}{\text{displacement}} = -\frac{YIk_j^3 T_j}{l^3}. \tag{50}$$

The torque is destabilising (positive feedback) and the force is restoring (negative feedback). One way to realise these conditions is shown in figure 7. The spring constant λ (force/displacement) must be

$$\lambda = YIk_j^3 T_j/2l^3 \simeq YI(j+\tfrac{1}{4})^3\pi^3/2l^3 \tag{51}$$

and the weights W and rod length d_0 must satisfy

$$Wd_0 = YIk_j/2lT_j \approx YI(j+\tfrac{1}{4})\pi/2l. \tag{52}$$

The degenerate modes have frequencies

$$\omega_j = \frac{k_j^2}{l^2}\left(\frac{YI}{\rho}\right)^{1/2} \approx \frac{(j+\frac{1}{4})^2\pi^2}{l^2}\left(\frac{YI}{\rho}\right)^{1/2}. \tag{53}$$

An experimental test of this theory could be made by periodically forcing the beam (for example by vibrating the clamp at $x = 0$) and determining the frequencies ω_j as resonances of the response function (which could be the RMS displacement at $x = l$). By varying λ and d_0 the diabolical structure of the spectrum could be studied. Because of mechanical dissipation it is probable that the Q of the resonances would be such that only the lowest degeneracies could be detected. The precise linear combination of modes that would be excited exactly at the degeneracies would depend delicately on how the beam was forced.

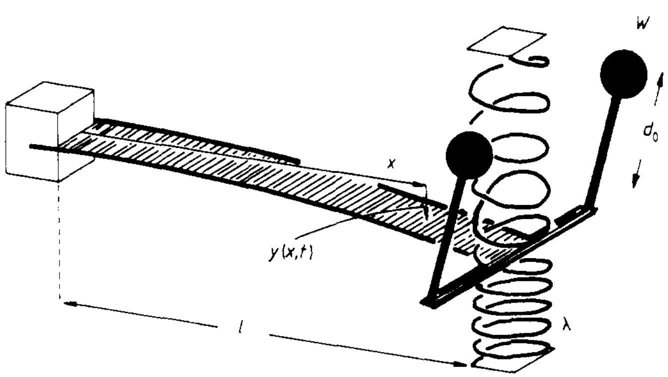

Figure 7. Vibrating beam with feedback boundary conditions which can cause degeneracy.

6. Discussion

We have shown that the energy levels of a quantal system inhabiting a finite line segment can be degenerate, provided the Hamiltonian has a higher than quadratic momentum dependence and thereby escapes the Sturm–Liouville straitjacket. The degeneracies can occur between states of the same symmetry, and are generic in both codimension and local diabolicity.

In our example the parameters appear in the boundary conditions, but there is nothing special (except ease of calculation) about this choice, and we could for example have worked with $H = p^4 + V(x;\, a, b)$ where $-\infty < x < \infty$ and (in the real case) V is a potential depending on two parameters a and b.

Another phenomenon which occurs generically but is forbidden in quadratic Hamiltonians in one dimension is degeneracy of states with different symmetry, which requires only one parameter to be varied. This also can be made to happen with quartic one-dimensional Hamiltonians. One example is the analogue of a double well, where the potential is $V = a\,\delta(x)$ and $\psi(\pm 1) = \psi'(\pm 1) = 0$; this has degeneracies as the 'barrier' height a varies.

Acknowledgments

We thank Dr M Wilkinson for suggesting that Hamiltonians quartic in the momentum might display degeneracies and Dr J H Hannay for helpful discussions. This work was supported (for RJM) by the Consejo Nacional de Ciencia *y* Tecnologia, Mexico, and not by any military agency.

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